Inequality 4

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Q> $a, b \in \mathbb{R}$ and $0 < a \leq b \leq 1$. Prove that

$$0 \leq \left(\frac{a}{1+b} + \frac{b}{1+a}\right) \leq 1$$

Ans:- $0 < b \leq 1$ $\qquad$ $0 < a \leq 1$

$1 < 1+b \leq 2$ $\qquad$ $1 < 1+a \leq 2$

$a < 1+b$ $\qquad$ $0 < \frac{b}{1+a} < 1$

$0 < \frac{a}{1+b} < 1$

$1+b \geq 1+a$

$\frac{1}{1+b} \leq \frac{1}{1+a}$

$\frac{a}{1+b} \leq \frac{a}{1+a}$ $\Rightarrow$ $\frac{a}{1+b} + \frac{b}{1+a} \leq \frac{a}{1+a} + \frac{b}{1+a} = \frac{a+b}{1+a} \leq 1$

$a \leq 1$, $b \leq 1$

$\Rightarrow a+b \leq 1+a$

$\Rightarrow \frac{a+b}{1+a} \leq 1$

$$0 \leq \frac{a}{1+b} + \frac{b}{1+a} \leq 1$$

Q> $a, b \in \mathbb{R}$ and $0 < a \leq b \leq 1$. Prove that $0 \leq ab^2 - ba^2 \leq \frac{1}{4}$

Ans:- $b \leq 1 \Rightarrow b^2 \leq b \Rightarrow -b^2 \geq -b$

$$ab^2 + (-ba^2) \leq ab^2 + (-b^2a^2) = b^2(a - a^2) \leq a - a^2 \leq \frac{1}{4}$$

$[\text{as } b^2 \leq 1]$

$$a - a^2 = -\left(\tfrac{1}{2}\right)^2 + 2 \times \tfrac{1}{2}a - a^2 + \frac{1}{4}$$

$$= \frac{1}{4} - \left(\frac{1}{2} - a\right)^2 \leq \frac{1}{4} \quad \ldots \left[\text{as } \left(\tfrac{1}{2} - a\right)^2 \geq 0\right]$$

$$ab^2 - ba^2 \leq \frac{1}{4}$$

$a > 0$

$-a < 0$ $\qquad$ $\frac{1}{4} - a < \frac{1}{4}$

$1 - a < 1$

$[x]$ is the greatest integer that is less than or equal to x.

$\{x\}$ is $x - [x]$ is the fractional part of x.

$\{-2.5\} = 0.5 = -2.5 - (-3)$

$\{2.5\} = 2.5 - 2 = 0.5$

As $x \geq [x]$ we get $\{x\} \geq 0$ and $\{x\} \leq 1$

$\Rightarrow \quad 0 \leq \{x\} < 1$

Q) Prove that for any positive integer n, the fractional part of $\sqrt{4n^2+n}$ is smaller than $\frac{1}{4}$.

i.e., $\{\sqrt{4n^2+n}\} < \frac{1}{4}$

Hint:- $4n^2 < 4n^2+n < 4n^2+4n+1$

Ans:- $2n < \sqrt{4n^2+n} < 2n+1$

$[\sqrt{4n^2+n}] = 2n$

$4n^2+n < 4n^2+n+\frac{1}{16}$

$\Rightarrow \sqrt{4n^2+n} < 2n+\frac{1}{4} \Rightarrow \{\sqrt{4n^2+n}\} < \frac{1}{4}$

Home Work:- Let a, b, c be positive real numbers such that, $abc = 1$

Prove that,

$$\frac{ab}{a^5+b^5+ab} + \frac{bc}{b^5+c^5+bc} + \frac{ca}{c^5+a^5+ca} \leq 1$$

* Quadratic Functions:-

Case I: $a > 0$,

$$ax^2 + 2bx + c^2 = a\left(x^2 + \frac{2b}{a}x + \frac{b^2}{a^2}\right) + c - \frac{b^2}{a}$$

$$= a\left(x + \frac{b}{a}\right)^2 + c - \frac{b^2}{a}$$

We know, $a\left(x + \frac{b}{a}\right)^2 \geq 0$, when $x = -\frac{b}{a}$ it achieves minimum which is $c - \frac{b^2}{a}$

Case II: $a < 0$

$a\left(x + \frac{b}{a}\right)^2 \leq 0$; when $x = -\frac{b}{a}$ it achieves maximum which is $c - \frac{b^2}{a}$

Case III: $a = 0$,

$ax^2 + 2bx + c = 2bx + c$ → min $= -\infty$, max $= \infty$ } so no bound

Example:- If x, y are positive numbers with $x + y = 2a$, then the product xy is maximal when $x = y = a$

Solution:- $x + y = 2a \Rightarrow y = 2a - x$

$xy = x(2a - x) = -x^2 + 2ax = a_0x^2 + 2b_0x + c_0$ (let)

$a_0 < 0 \Rightarrow$ Maximum is when $x = -\frac{b_0}{a_0} = -\frac{a}{-1} = a$

$y = 2a - x = 2a - a = a$

$\Rightarrow$ Maximum when $x = y = a$

Q) If x, y are positive numbers with $xy = 1$, then the sum $x+y$ is minimal when $x = y = 1$.

Ans:— $xy = 1$, $x = \frac{1}{y}$

$$y + \frac{1}{y} = (\sqrt{y})^2 + \left(\frac{1}{\sqrt{y}}\right)^2$$

$$= (\sqrt{y})^2 - 2 \times \sqrt{y} \times \frac{1}{\sqrt{y}} + \left(\frac{1}{\sqrt{y}}\right)^2 + 2$$

$$= \left(\sqrt{y} - \frac{1}{\sqrt{y}}\right)^2 + 2$$

Minimum when $\sqrt{y} - \frac{1}{\sqrt{y}} = 0 \Rightarrow \sqrt{y} = \frac{1}{\sqrt{y}} \Rightarrow x = y$

$xy = 1 \Rightarrow y^2 = 1 \Rightarrow y = 1 = x \Rightarrow x = y = 1$ gives minimum

- For any number $x > 0$ we get,
  $$x + \frac{1}{x} \geq 2$$
- For any number $x < 0$ we get,
  $$x + \frac{1}{x} \leq -2$$

$\Bigg\}$ Homework:— Verify this

Q) If $a, b > 0$ then $\frac{a}{b} + \frac{b}{a} \geq 2$ and the equality holds if and only if $a = b$.

If $a = b$ then $\frac{a}{b} + \frac{b}{a} = 2$

Ans:— $a^2 - 2ab + b^2 \geq 0$

$a^2 + b^2 \geq 2ab$

If, $\frac{a}{b} + \frac{b}{a} = 2$

$\Rightarrow a^2 + b^2 = 2ab$

Ans: — $a^2 - 2ab + b^2 \geq 0$

$$a^2 + b^2 \geq 2ab$$

$$\frac{a^2+b^2}{ab} \geq 2$$

$$\frac{a}{b} + \frac{b}{a} \geq 2$$

$$\Rightarrow a^2 + b^2 = 2ab$$

$$\Rightarrow a^2 - 2ab + b^2 = 0$$

$$\Rightarrow (a-b)^2 = 0$$

$$\Rightarrow a = b$$